Linear Transformations That Preserve Immanants*

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ABSTRACT

We characterize the linear transformations on matrices that preserve the immanants, i.e., $d_\chi(T(X)) = d_\chi(X)$ for all X, where χ is an irreducible character of degree greater than one on S_n. We present a necessary and sufficient condition for T to be invertible if it preserves the generalized matrix function d_c, and prove that the preservers of immanants are always nonsingular.

1. INTRODUCTION

Let S_n be the symmetric group of degree n. Let F be an arbitrary field, and c a function, not identically zero, from S_n into F. If $X = [x_{ij}]$ is an $n \times n$ matrix over F, the generalized Schur function $d_c(X)$ is defined by

$$d_c(X) = \sum_{\sigma \in S_n} c(\sigma) \prod_{i=1}^{n} x_{i\sigma(i)}.$$

Let c coincide with a character χ of a subgroup H of S_n and be zero in $S_n \setminus H$. We denote $d_c(X)$ by $d_\chi^H(X)$ and say that it is the generalized matrix function associated with H and χ.

*This work was carried out within the activities of the Centro de Álgebra da Universidade de Lisboa, INIC.

655 Avenue of the Americas, New York, NY 10010 0024-3795/94/$7.00

The characterization of linear operators on matrix spaces that preserve certain functions has been studied since 1897, when Frobenius [3] gave the form of linear transformations on $n \times n$ complex matrices that hold the determinant fixed, i.e. all those linear transformations T such that

$$\det T(X) = \det X$$

for all X. This result was re-proved later on by Henryk Minc [7] using elementary matrix methods.

In 1960 Marvin Marcus and F. C. May [5] characterized those linear transformations on matrices which leave the permanent unaltered, i.e.

$$\operatorname{per} T(X) = \operatorname{per} X$$

for all X. Peter Botta [2] proved the same result in a somewhat more direct way.

In [8] G. N. de Oliveira presents the following problem: Let $\mathscr{M}_n(F)$ be the linear space of n-square matrices with elements in F. Find all the linear operators $T : \mathscr{M}_n(F) \to \mathscr{M}_n(F)$ that preserve a certain Schur function, i.e., such that

$$d_\chi^H(T(A)) = d_\chi^H(A) \qquad \text{for every} \quad A \in \mathscr{M}_n(F).$$

He gave some examples of preservers of Schur functions.

Here we study the preservers of those generalized matrix functions for which $H = S_n$ and χ is an irreducible character. In this case, d_χ^H, which we abbreviate to d_χ, is called an *immanant*. Obviously det and per are special cases of immanants, and as in both cases the problem is already studied, we shall consider χ as an irreducible nonlinear character. So we shall study which linear transformations on matrices preserve the immanants, i.e.

$$d_\chi(T(X)) = d_\chi(X) \tag{1.1}$$

for all X, and χ an irreducible character of degree greater than one on S_n.

One preliminary question in such studies is whether T must necessarily be invertible. Using a result of William C. Waterhouse [10], we state a necessary and sufficient condition for T to be invertible, if it preserves d_c, and prove that the immanant preservers are always nonsingular.

The irreducible characters on S_n are in bijective correspondence with the ordered partitions of n. We define a partition α of n as $\alpha = (\alpha_1, \dots, \alpha_r)$

where the α_i's are integers, $\alpha_1 \geqslant \cdots \geqslant \alpha_r \geqslant 1$, and $\alpha_1 + \cdots + \alpha_r = n$. The α_i's are called the *parts* of α; the number of parts is the *length* of α, denoted $L(\alpha)$. Each partition is related to a Young diagram, which consists of $L(\alpha)$ left justified rows of boxes. The number of boxes in the ith row is α_i.

The immanants which correspond to partitions of the form $(k, 1, \dots, 1)$ are called single-hook immanants. Denote the corresponding character by χ_k and the corresponding immanant by d_k. For example, if we let $\text{fix}(\sigma)$ denote the number of fixed points of $\sigma \in S_n$, then

$$\begin{aligned}
\chi_n(\sigma) &= 1, \qquad \text{the principal character,} \\
\chi_{n-1}(\sigma) &= \text{fix}(\sigma) - 1, \\
\chi_2(\sigma) &= \epsilon(\sigma)[\text{fix}(\sigma) - 1], \\
\chi_1(\sigma) &= \epsilon(\sigma), \qquad \text{the alternating character.}
\end{aligned}$$

The immanants d_n and d_1 are therefore the familiar permanent and determinant functions. For $\chi_{n-2}(\sigma), \dots, \chi_3(\sigma)$ no simple expression is known.

In Section 2 we introduce some notation and mention some results that will be used later. In Section 3 we state our main results, and in Section 4 we give the proofs.

2. PRELIMINARIES

For any $n \times n$ matrix $X \in \mathcal{M}_n(F)$, $X[i_1, \dots, i_r | j_1, \dots, j_r]$ will denote the submatrix of X formed by retention of rows $i_1, \dots, i_r$ and columns $j_1, \dots, j_r$. We let E_{ij} be the $n \times n$ matrix with 1 in position (i, j) and 0 elsewhere.

If $\sigma \in S_n$, we denote by $P(\sigma)$ the permutation matrix $(n \times n)$ whose (i, j) entry is

$$P(\sigma)_{ij} = \delta_{i\sigma(j)}, \qquad i, j \in \{1, \dots, n\}.$$

The σ-diagonal of the matrix X is the set $\{x_{i\sigma(i)} : i = 1, \dots, n\}$.

If $C \in \mathcal{M}_n(F)$ and $X \in \mathcal{M}_n(F)$, we define the Hadamard product of C and X to be the matrix $Y = C * X \in \mathcal{M}_n(F)$ given by $y_{ij} = c_{ij} x_{ij}$ $(i, j = 1, \dots, n)$.

Using the definition of the radical of a function given in [10], we shall say that a matrix $Y \in \mathcal{M}_n(F)$ is in the radical of d_c if $d_c(aY + X) = d_c(X)$ for all $a \in F$ and $X \in \mathscr{M}_n(F)$. The radical R of d_c is obviously a subspace of $\mathscr{M}_n(F)$, and Proposition 1 and a consequence of Proposition 2 of [10] can be easily reformulated as:

PROPOSITION 1. *The radical of d_c is zero if and only if all the linear maps $T : \mathscr{M}_n(F) \to \mathscr{M}_n(F)$ satisfying $d_c(T(X)) = d_c(X)$ for all X in $\mathscr{M}_n(F)$ are invertible.*

PROPOSITION 2. *If the linear transformation $T : \mathscr{M}_n(F) \to \mathscr{M}_n(F)$ preserves d_c, then $T(R) \subseteq R$.*

We may choose a complement V_1 of R so that $\mathscr{M}_n(F) = V_1 \oplus R$, and if we choose a basis of $\mathscr{M}_n(F)$ adapted to this decomposition, by Proposition 2 the linear transformations T preserving d_c have matrices of size $n^2 \times n^2$ of the form

$$\begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}. \tag{2.1}$$

3. DEFINITIONS AND MAIN RESULTS

Let N be the set of pairs (i, j), where $i, j \in \{1, \dots, n\}$, such that for every $\sigma \in S_n$, if $\sigma(i) = j$ then $c(\sigma) = 0$. In other words, N is just the set of pairs (i, j) such that $d_c(X)$ does not depend on x_{ij}.

Let $\mathscr{N}$ be the subspace generated by those matrices E_{ij} such that $(i, j) \in N$, and $\mathscr{N} = (0)$ if $N = \varnothing$.

THEOREM 1. *Let T be a linear transformation of $\mathscr{M}_n(F)$ into itself, and suppose $d_c(T(X)) = d_c(X)$ for all $X \in \mathscr{M}_n(F)$. Then $N = \varnothing$ iff all the transformations T are nonsingular. If $N \neq \varnothing$, then T is nonsingular iff the restriction of T to $\mathscr{N}$ is nonsingular.*

If $H = S_n$ and χ is an irreducible character of S_n, then $N = \varnothing$. So we have

COROLLARY. *If T is an immanant preserver, then T is nonsingular.*

Now we shall characterize those linear transformations on matrices which leave an immanant unaltered. We shall separate the cases $n = 3$ and $n \geqslant 4$. From now on the underlying field is $\mathbb{C}$, the complex numbers.

THEOREM 2. *Let $n \geqslant 4$. A linear transformation T of $\mathscr{M}_n(\mathbb{C})$ into itself preserves the immanant corresponding to the character χ of degree greater* *than one iff either*

(i) $T(X) = C * P(\tau_1)XP(\tau_2)$ *or*
(ii) $T(X) = C * P(\tau_1)X^TP(\tau_2)$ *and*

$$\chi(\pi)\prod_{i=1}^{n} c_{i\pi(i)} = \chi(\tau_2\pi\tau_1) \qquad \textit{for all} \quad \pi \in S_n.$$

Let us remark that if $n = 3$, we have only to study which T preserve the second immanant d_2, and we have

$$d_2(X) = 2x_{11}x_{22}x_{33} - x_{12}x_{23}x_{31} - x_{13}x_{21}x_{32}.$$

In this case the character is nonzero just in the alternating group A_3, and for each (i, j) with $i, j = 1, 2, 3$, there is just one $\sigma \in A_3$ such that $\sigma(i) = j$.

Let us recall that a *σ-diagonal* of a matrix X is the set of the elements $x_{i\sigma(i)}$, where σ is an arbitrary permutation.

If we define a bijective function $f: A_3 \to A_3$, we have $\chi(f(\sigma)) \neq 0$ for all $\sigma \in A_3$.

THEOREM 3. *A linear transformation T of $M_3(\mathbb{C})$ into itself preserves the second immanant iff*

$$T(X) = C * Y,$$

where Y is a matrix obtained from X by at least one of the following operations:

(1) *permutation of the elements in each σ = diagonal of X if $\sigma \in A_3$,*
(2) *permutation of the σ-diagonals if $\sigma \in A_3$;*

and if we represent (2) by a bijective function f from A_3 onto A_3, the entries c_{ij} of the matrix C satisfy the condition

$$\prod_{i=1}^{3} c_{if(\sigma)(i)} = \frac{\chi(\sigma)}{\chi(f(\sigma))} \qquad \textit{for all} \quad \sigma \in A_3.$$

4. PROOFS

To prove Theorem 1, it is enough to remark that the subspace $\mathscr{N}$ is just the radical R of the function d_c, as we show in the following proposition.

PROPOSITION 3. *The subspace $\mathscr{N}$ is the radical of d_c.*

Proof. First let us prove that $\mathscr{N}$ is a subspace of R. If $N = \varnothing$, then by definition $\mathscr{N} = 0$, and thus a subspace of R. If $N \neq \varnothing$, for every matrix E_{ij} such that $(i, j) \in N$, we see that $d_c(xE_{ij} + B) = d_c(B)$ for all $x \in F$ and $B \in \mathscr{M}_n(F)$. This implies that $\mathscr{N}$ is a subspace of the radical R.

Consider now a matrix $A = \sum_{i,j} a_{ij} E_{ij}$ belonging to the radical of d_c; by definition

$$d_c(xA + B) = d_c(B) \qquad \text{for all} \quad x \in F \text{ and } B \in \mathscr{M}_n(F). \qquad (4.1)$$

For any $(i, j) \notin N$ there is $\sigma \in S_n$ such that $\sigma(i) = j$ and $c(\sigma) \neq 0$. Let us fix (i, j) and choose B defined by

$$\begin{aligned} b_{i\sigma(i)} &= 0 \\ b_{k\sigma(k)} &= 1 - xa_{k\sigma(k)} && \text{if} \quad k \neq i \\ b_{pq} &= -xa_{pq} && \text{for all the other entries of } B, \end{aligned}$$

and let us compute $d_c(xA + B)$. We have $d_c(xA + B) = c(\sigma)a_{ij}x$, which must be equal to $d_c(B)$, by (4.1). As $d_c(B)$ is a polynomial in x without terms of degree 1, and σ was chosen so that $c(\sigma) \neq 0$, if the equality (4.1) holds, then $a_{ij} = 0$ whenever $(i, j) \notin N$. This means that R is a subspace of $\mathscr{N}$. ■

COROLLARY. *The radical of d_c is the zero subspace iff $N = \varnothing$.*

Proof of Theorem 1. Assume that $N \neq \varnothing$. As $\mathscr{N} = R$, if we rearrange the basis $\{E_{11}, \dots, E_{1n}, E_{21}, \dots, E_{nn}\}$ so that the last vectors E_{ij} belong to R, then by Proposition 2 the matrix of T has the form (2.1). If S is a linear transformation with matrix

$$\begin{bmatrix} T_{11} & 0 \\ * & * \end{bmatrix}$$

($*$ unspecified), we have $d_c(T(X)) = d_c(S(X))$, because neither of these values depends on the entries in the positions (i, j) whith $(i, j) \notin N$. So if we define a linear transformation $\tilde{T}$ with matrix

$$\begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix},$$

we have $d_c(\tilde{T}(X)) = d_c(T(X))$ and consequently $d_c(\tilde{T}(X)) = d_c(X)$ for all X. If we suppose that $\det T_{11} = 0$, then there is a nonzero vector $X_0 \in V_1$ such that $\tilde{T}(X_0) = 0$, and we have

$$\begin{aligned} d_c(aX_0 + Y) &= d_c\big(\tilde{T}(aX_0 + Y)\big) \\ &= d_c\big(a\tilde{T}(X_0) + \tilde{T}(Y)\big) \\ &= d_c\big(\tilde{T}(Y)\big) = d_c(Y) \end{aligned}$$

for all $a \in F$, $Y \in \mathscr{M}_n(F)$, and this means that $X_0 \in R$, a contradiction.

Consequently T is nonsingular iff $\det T_{22} \neq 0$, that is, the restriction of T to $\mathscr{N}$ is an isomorphism. ■

The proofs of Theorems 2 and 3 have been split into some steps.

The first step consists of characterizing the subset of $\mathscr{M}_n(\mathbb{C})$ that generalizes the one defined in [6]:

$$\mathscr{A} = \big\{A : \deg d_\chi(xA + B) \leqslant 1 \qquad \text{for every } B \in M_n(\mathbb{C})\big\},$$

where $\deg p(x)$ denotes the degree of the polynomial $p(x)$. If A belongs to $\mathscr{A}$, and T is an immanant preserver (therefore nonsingular), we have $\deg d_\chi(xT(A) + T(B)) \leqslant 1$, where $T(B)$ can be any matrix in $\mathscr{M}_n(\mathbb{C})$. Thus $T(A) \in \mathscr{A}$.

The second step consists in characterizing the image $T(A)$ of some particular matrices A belonging to $\mathscr{A}$. Let us consider $i \neq k$; in what follows we will denote by σ and τ two permutations of S_n related by $\tau = \sigma(i\,k)$ [$(i\,k)$ denotes the transposition that interchanges i and k]. So if $j = \sigma(i)$ and $h = \sigma(k)$ we can write

$$\begin{aligned} &\sigma(i) = j, && \tau(i) = h, \\ &\sigma(k) = h, && \tau(k) = j, \\ &\sigma(p) = \tau(p) && \text{for all} \quad p \neq i, k. \end{aligned} \tag{4.2}$$

LEMMA 1. *Let $A \in \mathscr{A}$ and $i \neq k$. Then for every σ in S_n and $\tau = \sigma(i\,k)$ we have*

$$\chi(\sigma)a_{i\sigma(i)}a_{k\sigma(k)} + \chi(\tau)a_{i\tau(i)}a_{k\tau(k)} = 0.$$

Proof. In the definition of $\mathscr{A}$ take as B the matrix defined by

$$B = \sum_{t \neq i, k} E_{t\sigma(t)},$$

and notice that the coefficient of x^2 in $d_\chi(xA + B)$ is $\chi(\sigma)a_{i\sigma(i)}a_{k\sigma(k)} + \chi(\tau)a_{i\tau(i)}a_{k\tau(k)}$. ■

To prove Theorem 3 we need some technical lemmas and consider several remarks.

In what follows, s and t are integers satisfying $1 \leqslant s \leqslant t \leqslant n$. If $\alpha \in S_n$, let $\alpha = \alpha_1 \alpha_2 \cdots \alpha_t$ be its disjoint cyclic decomposition. Let p_s be the length of the cycle α_s. Consequently $p_s \geqslant 1$ and $\Sigma_s p_s = n$. We will identify the conjugacy class of each permutation α with the partition $(p_1, p_2, \dots, p_t)$. We represent by $(p_1, 1^r)$ the same class of permutations as $(p_1, 1, \dots, 1)$, where the 1 appears $r = n - p_1$ times. If $r = 0$, then it is the same as the class (n).

Let D be the Young diagram representing the character $[m_1, m_2, \dots, m_{t_1}]$. The integers $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_{t_1}$ $(\geqslant 1)$ denote the number of boxes in each row, and the integers $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_{m_1}$ $(\geqslant 1)$ denote the number of boxes in each column of the diagram D. Note that $(t_1, t_2, \dots, t_{m_1})$ is the conjugated partition of the partition $(m_1, m_2, \dots, m_{t_1})$. In what follows we use the Murnaghan-Nakayama rule to calculate the values of the character.

LEMMA 2. *Let χ be an irreducible character of S_n of degree greater than one $(n \geqslant 3)$, associated with the partition $(m_1, m_2, \dots, m_{t_1})$. Let $(t_1, t_2, \dots, t_{m_1})$ be the conjugated partition, and let $q_1 = \max(m_1 - 1, t_1 - 1)$ and $q_2 = \min(m_1 - 1, t_1 - 1)$. Then:*

(1) *The value of χ is not zero in the following conjugacy classes:*

(i) $(q_1 + q_2 + 1, 1^{n-(q_1+q_2+1)})$
(ii) $(m_1 + t_1 - 1, m_2 + t_2 - 3, 1^{n-(m_1+t_1+m_2+t_2-4)})$
(iii) $(q_1 + 1, n - (q_1 + 1))$ *if* $m_1 + t_1 = n$,
(iv) $(q_1, q_2, 1)$ *if* $m_1 + t_1 = n + 1$.

(2) *The value of χ is zero in the following conjugacy classes:*

(i) *Conjugacy classes that contain a cycle of length larger than* $m_1 + t_1 - 1$,
(ii) $(q_1 + 1, q_2)$ *if* $m_1 + t_1 = n + 1$,
(iii) $(q_1 + q_2, 1)$ *if* $m_1 + t_1 = n + 1$.

Proof. Let us begin with the following remark: If D is the Young diagram associated with $[m_1, m_2, \dots, m_{t_1}]$, and $[t_1, t_2, \dots, t_{m_1}]$ is the conjugated diagram, then D has $m_1 + t_1 - 1$ boundary boxes, because in each row j there are $m_j - m_{j+1} + 1$ boundary boxes, except the last, which has m_{t_1}. If we have in a diagram that $m_1 + t_1 - 1 < n$, and we omit all the boundary boxes, the diagram so obtained has $m_2 + t_2 - 3$ boundary boxes.

If $q_1 = \max(m_1 - 1, t_1 - 1)$ and $q_2 = \min(m_1 - 1, t_1 - 1)$, then $q_1 + q_2 + 1 = m_1 + t_1 - 1$, just the number of the boundary boxes.

(1): In classes (1)(i) and (1)(ii), if D has not a single-hook structure, after removing all boundary parts with lengths greater than 1, there remains a frame which one must reduce by cutting out single boxes only. The number of ways in which this can be done is just equal to the degree of the representation corresponding with the obtained diagram, which we denote by q. So certainly $q \neq 0$. So if π belongs to class (1)(i), then $\chi(\pi) = q(-1)^{t_1-1} \neq 0$, since there is only one way of omitting the $q_1 + q_2 + 1$ boundary boxes. And if π belongs to (1)(ii), then $\chi(\pi) = q(-1)^{(t_1-1)+(t_2-2)} \neq 0$. If D has a single-hook structure, both classes are the same, just the class (n), and obviously, if π belongs to this class, $\chi(\pi) = (-1)^{t_1-1}$.

In class (1)(iii), we have $m_1 + t_1 = n$, that is, the diagram D has just one box not in the boundary; then $q_1 + 1$ is just the maximum number of boxes we can omit, if we do not want to omit all the boundary boxes. If $q_1 > q_2$, there is a unique way of omitting $q_1 + 1$ boundary boxes, and consequently, if π belongs to class (1)(iii), then $\chi(\pi) = (-1)^{t_1-2} \neq 0$ or $\chi(\pi) = (-1)^{t_1} \neq 0$. If $q_1 = q_2$, there are two ways in which the diagram D can be reduced to zero, but the number of the vertical steps in one way is t_1, and in the other is $t_1 - 2$, and thus these numbers are both even or both odd, and so if $\pi \in (q_1 + 1, n - (q_1 + 1))$ then $\chi(\pi) = 2$ or $\chi(\pi) = -2$.

In class (1)(iv), as $m_1 + t_1 - 1 = n$ (that is, the diagram D has a hook structure), if $\pi \epsilon (q_1, q_2, 1)$ and $q_1 > q_2$ then $\chi(\pi) = (-1)^{t_1-2} \neq 0$, and if $q_1 = q_2$ then $\chi(\pi) = 2(-1)^{t_1-2} \neq 0$.

(2): It is obvious that χ must be equal to zero in those classes, because in any case there are no regular boundary parts with the required lengths. ■

The following remarks are consequences of Lemma 2 and will be used later.

REMARK 1. Let $i \neq k$, both belonging to $\{1, \ldots, n\}$, and $n > 3$. There are permutations σ and τ satisfying

$$\sigma(i) = i, \quad \sigma(k) \neq k, \quad \tau = \sigma(ik) \quad \text{and} \quad \chi(\sigma) \neq 0, \quad \chi(\tau) = 0.$$

Since $\sigma(i) = i$, σ must contain at least a cycle with length one, and τ a cycle with length at least three. If $m_1 + t_1 = n + 1$, there are σ and τ belonging respectively to the classes $(q_1, q_2, 1)$ and $(q_1 + 1, q_2)$. If $m_1 + t_1 < n + 1$, we can choose σ in the class $(q_1 + q_2 + 1, 1^{n-(q_1+q_2+1)})$ and τ in the class $(q_1 + q_2 + 2, 1^{n-(q_1+q_2+2)})$.

REMARK 2. Let $i \neq k$, both belonging to $\{1, \ldots, n\}$, and $n \geqslant 3$. There are permutations σ and τ satisfying

$$\sigma(i) = i, \quad \sigma(k) \neq k, \quad \tau = \sigma(ik) \quad \text{and} \quad \chi(\tau) \neq 0.$$

We can choose τ in the class $(m_1 + t_1 - 1,\ m_2 + t_2 - 3,\ 1^{n-(m_1+t_1-1+m_2+t_2-3)})$, because $m_1 + t_1 - 1 \geqslant 3$, and $\chi(\tau) \neq 0$ by Lemma 2. Note that if $m_1 + t_1 = n + 1$ this class is just (n).

REMARK 3. Let $n \geqslant 4$, and i, j, k, h be pairwise distinct. Assume $\chi \neq \chi_2, \chi_{n-1}$. There are permutations σ and τ satisfying

$$\sigma(i) = j, \quad \sigma(k) = h, \quad \tau = \sigma(ik) \quad \text{and} \quad \chi(\sigma) \neq 0, \quad \chi(\tau) = 0.$$

Firstly notice that we can choose σ with at least two distinct cycles (one containing i and j, the other containing k and h), and τ with one cycle of length at least four (containing i, j, k, h). If the diagram D has two or more boxes not in the boundary ($n \geqslant 5$), that is, $m_2 + t_2 - 3 \geqslant 2$, then we can choose σ in the class $(m_1 + t_1 - 1, m_2 + t_2 - 3, 1^r)$, where $r = n - (m_1 + t_1 - 1 + m_2 + t_2 - 3)$, and τ in the class $(m_1 + t_1 - 1 + m_2 + t_2 - 3, 1^r)$, because $m_1 + t_1 - 1 \geqslant m_2 + t_2 - 3 \geqslant 2$ and $m_1 + t_1 - 1 + m_2 + t_2 - 3 \geqslant 4$, and χ never vanishes in the first class and is equal to zero in the second.

If the diagram D has only one box not in the boundary ($n \geqslant 4$), i.e., $m_2 + t_2 - 3 = 1$, then as $q_1 + 1 \geqslant n - (q_1 + 1) \geqslant 2$, we can choose σ in the class $(q_1 + 1, n - (q_1 + 1))$ and τ in the class (n), and they satisfy the required conditions.

If the diagram D has a hook structure, then $m_2 + t_2 = 2$. As by assumption the elements i, j, k, h are pairwise distinct and $\chi \neq \chi_2$ and $\chi \neq \chi_{n-1}$, then $n > 4$, and we can have σ in the class $(q_1, q_2, 1)$ and τ in the class $(q_1 + q_2, 1)$ or the contrary, because $q_1 \geqslant q_2 \geqslant 2$ and $q_1 + q_2 \geqslant 4$. As we saw in Lemma 2, χ never vanishes in the first class and is zero in the second.

REMARK 4. Let $n \geqslant 4$, and i, j, k, h be pairwise distinct. Let χ be either χ_2 or χ_{n-1}. There are permutations σ and τ satisfying

$$\sigma(i) = j, \quad \sigma(k) = h, \quad \text{and} \quad \tau = \sigma(ik)$$

and

$$\chi(\sigma) = \chi(\tau) \neq 0 \qquad \text{in case} \quad \chi = \chi_{n-1}$$

or

$$\chi(\sigma) = -\chi(\tau) \neq 0 \quad \text{in case} \quad \chi = \chi_2.$$

In order to achieve $\chi(\sigma) \neq 0$, it is sufficient to define σ so that the number of fixed points will be different from 1. This is always possible.

In analogy with [2], let us characterize the matrices of $\mathscr{A}$. Let $R_{(i)}$ ($R^{(i)}$) be the subspace of $\mathscr{M}_n(\mathbb{C})$ consisting of all matrices that have nonzero entries only in row i (column i).

LEMMA 3. *Let $n > 3$. A matrix A belongs to $\mathscr{A}$ if and only if it is in one of the following cases:*

(i) *A belongs to $R_{(i)}$ ($R^{(i)}$) for some i.*

(ii) *The nonzero elements are in the submatrix $A[i, h|i, h]$, and $\chi(\sigma) a_{ii}a_{hh} + \chi(\tau)a_{ih}a_{hi} = 0$ for every σ and τ satisfying $\sigma(i) = i$, $\sigma(h) = h$, and $\tau = \sigma(ih)$.*

(iii) *$\chi = \chi_2$, and there are complementary sets of indices $\{i_1, \dots, i_p\}$, $\{j_1, \dots, j_q\}$, such that the nonzero elements are in $A[i_1, \dots, i_p|j_1, \dots, j_q]$ and the rank of the matrix is one.*

(iv) *$\chi = \chi_{n-1}$, the nonzero elements are in a 2×2 submatrix $A[u, v|r, s]$, and the permanent of this submatrix is zero.*

Proof of Lemma 3. Let $A \in \mathscr{A}$. We have to consider several cases and subcases, which we list:

(1) The matrix has a nonzero element in the principal diagonal, $a_{i_0 i_0}$ say.

(1.1) The elements on row i_0 are zero except $a_{i_0 i_0}$.

(1.2) Besides $a_{i_0 i_0}$ there is another nonzero element on row i_0.

(2) All the principal elements are zero, and there is a nonzero nonprincipal element, $a_{i_0 h_0}$ say.

(2.1) $\chi \neq \chi_2, \chi_{n-1}$.

(2.1.1) $a_{h_0 i_0} \neq 0$.

(2.1.2) $a_{h_0 i_0} = 0$.

(2.2) $\chi = \chi_2$.

(2.3) $\chi = \chi_{n-1}$.

We examine all the cases now.

(1): Let k, h, i_0 be pairwise distinct. There are permutations σ and τ such that $\sigma(i_0) = i_0$, $\sigma(k) = h$, $\tau = \sigma(i_0 k)$, and $\chi(\sigma) \neq 0$ and $\chi(\tau) = 0$ (Remark 1). By Lemma 1 we have that

$$\chi(\sigma) a_{i_0 i_0} a_{kh} + \chi(\tau) a_{i_0 h} a_{k i_0} = 0.$$

Since $a_{i_0 i_0} \neq 0$, it follows that $a_{kh} = 0$ for all $k, h \in \{1, \dots, n\}$, where k, h, i_0 are pairwise distinct.

(1.1): Take $s \neq i_0$. If σ is the identity and τ the transposition $(i_0\ s)$, then from Lemma 1 we obtain

$$\chi(\sigma) a_{i_0 i_0} a_{ss} + \chi(\tau) a_{i_0 s} a_{s i_0} = 0. \tag{4.3}$$

Since $a_{i_0 s} = 0$, we have $a_{ss} = 0$ for all $s \neq i_0$. It follows that $A \in R^{(i_0)}$.

(1.2): Let $a_{i_0 h} \neq 0$, where $h \neq i_0$. Using once more Lemma 1 and Remark 2, we can choose permutations σ and τ in S_n such that $\sigma(i_0) = i_0$, $\sigma(k) = h$, $\tau = \sigma(i_0\ k)$, and $\chi(\tau) \neq 0$. Since $a_{i_0 h} \neq 0$ and $a_{kh} = 0$, we must have $a_{k i_0} = 0$ for all $k \neq i_0, h$.

Choose $s \neq i_0, h$. Take $\sigma = \mathrm{id}$, $\tau = (i_0 s)$; by Lemma 1 we have once more (4.3). Since $a_{s i_0} = 0$ for all $s \neq i_0, h$, then $a_{ss} = 0$ for all $s \neq i_0, h$.

Besides $a_{i_0 i_0}$ and $a_{i_0 h}$ there may be more nonzero elements on row i_0. Assume $a_{i_0 t} \neq 0$, $t \neq i_0, h$. We have seen that $a_{i_0 h} \neq 0$ implies

(i) column i_0 equal to zero, except $a_{i_0 i_0}$ and $a_{h i_0}$, and
(ii) principal elements equal to zero, except $a_{i_0 i_0}$ and a_{hh}.

In the same way, $a_{i_0 t} \neq 0$ $(t \neq i_0, h)$ implies that column i_0 is equal to zero except $a_{i_0 i_0}, a_{t i_0}$. So $a_{h i_0} = 0$. It also implies that all principal elements are equal to zero except $a_{i_0 i_0}, a_{tt}$. So $a_{hh} = 0$. Thus we conclude that $A \in R_{(i_0)}$. If however the only nonzero elements on row i_0 are $a_{i_0 i_0}$ and $a_{i_0 h}$, it is clear that all the elements of the matrix are zero except those in the submatrix $A[i_0\ h | i_0\ h]$. By Lemma 1, if σ fixes i_0 and h and $\tau = \sigma(i_0\ h)$, then

$$\chi(\sigma) a_{i_0 i_0} a_{hh} + \chi(\tau) a_{i_0 h} a_{h i_0} = 0.$$

(2): Take $k \neq i_0, h_0$. By Remark 2, there are σ and τ such that $\sigma(i_0) = i_0$, $\sigma(k) = h_0$, $\tau = \sigma(i_0\ k)$, and $\chi(\tau) \neq 0$. By Lemma 1

$$\chi(\sigma) a_{i_0 i_0} a_{k h_0} + \chi(\tau) a_{i_0 h_0} a_{k i_0} = 0.$$

It follows that $a_{k i_0} = 0$ for all $k \neq i_0, h_0$.

Now choose $j \neq i_0, h_0$. By Remark 2 there are σ and τ such that $\sigma(h_0) = h_0$, $\sigma(i_0) = j$, $\tau = \sigma(i_0\, h_0)$, and $\chi(\tau) \neq 0$. By Lemma 1

$$\chi(\sigma) a_{h_0 h_0} a_{i_0 j} + \chi(\tau) a_{h_0 j} a_{i_0 h_0} = 0.$$

This implies $a_{h_0 j} = 0$ for all $j \neq i_0, h_0$.

(2.1): Let k, j, i_0, h_0 be pairwise distinct. By Remark 3 there are σ and τ such that $\sigma(i_0) = h_0$, $\sigma(k) = j$, $\tau = \sigma(i_0\, k)$, and $\chi(\sigma) \neq 0$ and $\chi(\tau) = 0$. By Lemma 1

$$\chi(\sigma) a_{i_0 h_0} a_{kj} + \chi(\tau) a_{i_0 j} a_{k h_0} = 0;$$

this implies $a_{kj} = 0$ for all k, j, i_0, h_0 pairwise distinct.

(2.1.1): Assume now that $a_{h_0 i_0} \neq 0$. We saw that $a_{i_0 h_0} \neq 0$, implies

(i) column i_0 equal to zero, except $a_{h_0 i_0}$;
(ii) row h_0 equal to zero, except $a_{h_0 i_0}$.

Likewise $a_{h_0 i_0} \neq 0$ implies

(i) column h_0 equal to zero, except $a_{i_0 h_0}$;
(ii) row i_0 equal to zero, except $a_{i_0 h_0}$.

We conclude that all elements are zero except $a_{i_0 h_0}$ and $a_{h_0 i_0}$. They have by Lemma 1 to satisfy

$$\chi(\sigma) a_{i_0 h_0} a_{h_0 i_0} = 0$$

for every σ containing the transposition $(i_0\, h_0)$ in its cyclic decomposition.

(2.1.2): Assume now that $a_{h_0 i_0} = 0$. Choose k, h, i_0, h_0 pairwise distinct. By Remark 3, we can choose σ and τ such that $\sigma(i_0) = h_0$, $\sigma(k) = h$, $\tau = \sigma(i_0\, k)$, and $\chi(\sigma) = 0$ and $\chi(\tau) \neq 0$. By Lemma 1

$$\chi(\sigma) a_{i_0 h_0} a_{kh} + \chi(\tau) a_{i_0 h} a_{k h_0} = 0.$$

It follows that

$$a_{i_0 h} a_{k h_0} = 0 \tag{4.4}$$

for all k, i_0, h_0, h pairwise distinct.

Assume that for a certain $h_1 \neq h_0$, we have that $a_{i_0 h_1} \neq 0$. Then (4.4) implies that $a_{k h_0} = 0$ for all $k \neq i_0, h_0, h_1$. So on column h_0, except for $a_{i_0 h_0}$

and $a_{h_1h_0}$, all elements are zero. In fact $a_{h_1h_0}$ must be zero. Assume it is not. By Remark 2, we can choose σ and τ such that $\sigma(i_0)=h_0$, $\sigma(h_1)=h_1$, $\tau=\sigma(i_0h_1)$, and $\chi(\tau)\neq 0$. By Lemma 1

$$\chi(\sigma)a_{i_0h_0}a_{h_1h_1}+\chi(\tau)a_{i_0h_1}a_{h_1h_0}=0.$$

This implies $a_{h_1h_0}=0$. Clearly $A\in R_{(i)}$.

Only one case remains: for $h\neq h_0$, a_{i_0h} is zero. But clearly $A\in R^{(h_0)}$.

(2.2): Consider four indices $i\neq k$, $j\neq h$ (arbitrary). Choose σ and τ such that $\sigma(i)=j$, $\sigma(k)=h$, $\tau=\sigma(ik)$. By Lemma 1 we have that

$$\chi(\sigma)a_{ij}a_{kh}+\chi(\tau)a_{ih}a_{kj}=0.$$

If $i=j$ or $i=j$ and $k=h$, we have, as the principal elements are zero, that

$$\chi(\tau)a_{ih}a_{ki}=0$$

or

$$\chi(\tau)a_{ih}a_{hi}=0.$$

Since $\chi(\tau)=\chi_2(\tau)=\epsilon(\tau)[\mathrm{fix}(\tau)-1]$, we can choose τ so that $\chi(\tau)\neq 0$. Thus

$$a_{ih}a_{ki}=0$$

or

$$a_{ih}a_{hi}=0.$$

This shows that all the 2×2 submatrices which include principal elements have determinant equal to zero.

Now consider the indices i,k,j,h pairwise distinct. There are σ and τ such that $\sigma(i)=j$, $\sigma(k)=h$, $\tau=\sigma(ik)$, and $\chi(\sigma)=-\chi(\tau)\neq 0$, by Remark 4. We have then by Lemma 1 that

$$\chi(\sigma)a_{ij}a_{kh}+\chi(\tau)a_{ih}a_{kj}=0,$$

or

$$a_{ij}a_{kh}-a_{ih}a_{kj}=0.$$

Conclusion: all the 2×2 submatrices have determinant equal to zero. So A is of rank 1. Therefore there are $x_i, y_j \in \mathbb{C}$ such that

$$A = [x_i y_j].$$

Since all the principal elements are zero, there are complementary subsets of $\{1, \dots, n\}$, Ω and Ω', such that

$$x_i = 0 \qquad \text{if} \quad i \in \Omega,$$

$$y_j = 0 \qquad \text{if} \quad j \in \Omega'.$$

Thus the rows i for $i \in \Omega$ and columns j for $j \in \Omega'$ are zero. We can conclude that there are complementary sets $\{i_1, \dots, i_p\}$, and $\{j_1, \dots, j_q\}$ such that the nonzero elements of A are in the submatrix

$$A[i_1, \dots, i_p | j_1, \dots, j_q].$$

We have already shown that the matrix has rank 1.

(2.3): As in the preceding case, all the products

$$a_{ih} a_{ki} \quad \text{and} \quad a_{ih} a_{hi}$$

are zero. This shows that all 2×2 submatrices which include principal elements have permanent equal to zero.

Now consider the pairwise distinct indices i, k, j, h. There are σ and τ such that $\sigma(i) = j$, $\sigma(k) = h$, $\tau = \sigma(ik)$, and $\chi(\sigma) = \chi(\tau) \neq 0$, by Remark 4. We have then, by Lemma 1, that

$$\chi(\sigma) a_{ij} a_{kh} + \chi(\tau) a_{ih} a_{kj} = 0,$$

or

$$a_{ij} a_{kh} + a_{ih} a_{kj} = 0.$$

Conclusion: all the 2×2 submatrices have permanent equal to zero. By Lemma 3 in [2], either $A \in R_{(i_0)}$, $R^{(h_0)}$ for some h_0, or there exist indices $u < r$, $v < s$ such that per $A[u, r | v, s] = 0$ and all the other entries in A are zero.

Conversely, it is obvious that if A has one of the forms (i), (ii), (iii), or (iv), then $A \in \mathscr{A}$. ■

In [4] four general types of linear preservers are described. One of them, presented as Problem I, is concerned with the study of linear operators preserving certain functions, and another, called Problem II, is concerned with those linear operators preserving certain subsets. Very often these two problems can be related, as is the case here.

If T preserves d_χ, then T is nonsingular and $T(\mathscr{A}) \subseteq \mathscr{A}$. So we can regard the second step of the proof of Theorem 3 as a lemma on a problem of type II.

LEMMA 4. *Let T be nonsingular and satisfy $T(\mathscr{A}) \subseteq \mathscr{A}$. Then either*

(i) *There are π_1 and π_2 in S_n such that*

$$T(R_{(i)}) = R^{(\pi_1(i))} \quad \text{and} \quad T(R^{(k)}) = R_{(\pi_2(k))}$$

for $i, k = 1, \ldots, n$, or

(ii) *There are π_1, π_2 in S_n such that*

$$T(R_{(i)}) = R_{(\pi_1(i))} \quad \text{and} \quad T(R^{(k)}) = R^{(\pi_2(k))}$$

for $i, k = 1, \ldots, n$.

Proof. This lemma generalizes a result in [2], and the argument is similar to that of Botta.

Let $z_1, \ldots, z_n$ be indeterminates over $\mathbb{C}$, and consider the matrix over $\mathbb{C}[z_1, \ldots, z_n]$

$$Z = \sum_{t=1}^{n} z_t E_{it}.$$

Of course $T(Z)$ will also be a matrix over $\mathbb{C}[z_1, \ldots, z_n]$, and its (i, j) entry will be a linear form $L_{ij}(z_1, \ldots, z_n)$.

We will write $Z^{(0)}$ to mean a matrix over $\mathbb{C}$ obtained by assigning certain complex values to the indeterminates. Denote by (L) the system of linear equations obtained by equating to zero all the nonzero linear forms appearing in $T(Z)$. The system (L) cannot have a nonzero solution, as otherwise it would be possible to have $T(Z^{(0)}) = 0$ with $Z^{(0)} \neq 0$. But T is a nonsingular linear operator.

$T(Z)$ will have one of the forms (i), (ii), (iii), or (iv) of Lemma 3. If not, it would be possible to assign complex values to the indeterminates to obtain matrices $Z^{(0)}$ and $T(Z^{(0)})$ violating Lemma 3.

We show that $T(Z)$ must have form (i). Assume it has not, i.e., $T(Z)$ has one of the forms (ii), (iii), or (iv). We have to examine the following cases and subcases.

(1) $\chi \neq \chi_2$
  (1.1) $n > 4$
  (1.2) $n = 4$
    (1.2.1) χ is the character determined by the partition (2, 2).
    (1.2.2) χ is the character determined by the partition (3, 1).
(2) $\chi = \chi_2$.

Case (1): $T(Z)$ will be either of from (ii) or (iv).

Subcase (1.1): The total number of equations in (L) is at most 4, i.e., it is smaller than the number of indeterminates. The system (L) would have a nonzero solution, a contradiction.

Subcase (1.2): Firstly we examine (1.2.1). $T(Z)$ will have form (ii), and the entries of $T(Z)$ satisfy

$$\chi(\sigma)L_{ii}(z_1,\dots,z_4)L_{hh}(z_1,\dots,z_4) + \chi(\tau)L_{ih}(z_1,\dots,z_4)L_{hi}(z_1,\dots,z_4) = 0$$

for every σ and $\tau = \sigma(i\,h)$. Take as σ the identity permutation, and τ equal to a transposition. For the character we are considering, $\chi(\tau) = 0$. We conclude that one of the forms $L_{ii}(z_1,\dots,z_4)$ or $L_{hh}(z_1,\dots,z_4)$ is the zero form. Again the system (L) would have nonzero solution. Contradiction.

Now we examine (1.2.2). By Lemma 3 there are only four elements in $T(Z)$ that may be different from zero. These elements form a 2×2 matrix with permanent equal to zero. By the unique factorization theorem for $\mathbb{C}[z_1,\dots,z_n]$ we conclude that in the system (L) there are no more than two independent equations. So it will have a nonzero solution. Contradiction.

Finally let us look at case (2). The matrix $T(Z)$ has form (iii) and has rank 1. It is easy to see that because of this, the total number of independent equations in (L) is smaller than n. So it will have a nonzero solution. Contradiction.

So far we have proved that if $A \in R_{(i)}$, then $T(A)$ belongs to some $R_{(j)}$ or to some $R^{(k)}$. Assume $T(A) \in R_{(j)}$ (the other alternative would be treated in the same way). Let $B \in R_{(h)}$, $h \neq i$. Then $T(B)$ must belong to some $R_{(k)}$, and not to some $R^{(k)}$; otherwise T would not be onto. Moreover, k must
be different from j, for the same reason. Also, $T(R_{(i)})$ cannot be properly contained in $R_{(j)}$, so $T(R_{(i)}) = R_{(j)}$. If $A \in R_{(i)}$ and $T(A) \in R_{(k)}$ for some k, if we choose B in $R^{(j)}$, then $T(B)$ must belong to some $R^{(h)}$ (and not to some $R_{(h)}$) because T is one to one. The proof is complete. ∎

Proof of Theorem 2. The matrix E_{ij} belongs to $R_{(i)}$ and to $R^{(j)}$. Assume alternative (i) of Lemma 4 holds. For notational reasons we prefer to write π_1^{-1} instead of π_1. Then

$$T(E_{ij}) \in R^{\pi_1^{-1}(i)} \cap R_{\pi_2(j)},$$

and so there are complex numbers c_{ij} such that

$$T(E_{ij}) = c_{\pi_2(j)\pi_1^{-1}(i)} E_{\pi_2(j)\pi_1^{-1}(i)}.$$

Let X be an arbitrary matrix. Then $X = \sum_{i,j} a_{ij} E_{ij}$. We have

$$T(X) = \sum_{i,j} a_{ij} c_{\pi_2(j)\pi_1^{-1}(i)} E_{\pi_2(j)\pi_1^{-1}(i)},$$

or

$$T(X) = C * P(\pi_1) X^T P(\pi_2), \tag{4.5}$$

where $C = [c_{ij}]$. Setting $X = P(\pi)$, we obtain

$$\chi(\pi) \prod_{i=1}^{n} c_{i\pi(i)} = \chi(\pi_2 \pi \pi_1) \tag{4.6}$$

for every π in S_n.

Conversely, let (4.5) and (4.6) be satisfied. Computing $d_\chi(T(X))$, we see it is equal to $d_\chi(X)$. If in Lemma 4 we assume that alternative (ii) holds, we get alternative (i) of Theorem 2. ∎

Finally we will prove Theorem 3.

LEMMA 5. *Let $n = 3$. Then $\mathscr{A}$ consists precisely of those matrices which have at most one nonzero element in each σ-diagonal for every σ in A_3.*

Proof. Notice that A_3 consists of the identity and two cycles. If χ is of degree larger than one, $\chi(\phi) = \mathrm{fix}(\phi) - 1$. So $\chi(\phi)$ is not zero if and only if $\phi \in A_3$.

Let $A \in \mathscr{A}$. Now choose two distinct indices i and k. We show that one of $a_{i\sigma(i)}$, $a_{k\sigma(k)}$, for any σ in A_3, is zero. We have

$$\chi(\sigma) a_{i\sigma(i)} a_{k\sigma(k)} + \chi(\tau) a_{i\tau(i)} a_{k\tau(k)} = 0 \qquad \text{for every } \tau \text{ in } S_3. \tag{5.1}$$

Take $\tau = \sigma(i, k)$. If σ is the identity permutation, τ is a transposition and so $\chi(\tau) = 0$. If σ is a cycle, then $\tau \notin A_3$ (otherwise $(i\,k)$ would belong to A_3) and so $\chi(\tau) = 0$. In either case (5.1) gives

$$\chi(\sigma) a_{i\sigma(i)} a_{k\sigma(k)} = 0.$$

Since $\chi(\sigma) \neq 0$, one of the other factors must be zero.

Conversely, if for every σ in A_3 the σ-diagonal has at most one nonzero element, then clearly $A \in \mathscr{A}$. ∎

REMARK. Let us denote by $D(\pi)$ the set of positions covered by the diagonal $(1, \pi(1))$, $(2, \pi(2))$, $(3, \pi(3))$, and by $\mathscr{D}(\pi)$ the set of matrices generated by $\{E_{ij} : (i, j) \in D(\pi)\}$.

If $A \in \mathscr{A}$, it is obvious that the total number of nonzero entries cannot exceed 3, i.e., $h(A) \leqslant 3$. We have also that if $A \in R_{(i)}$, or $A \in R^{(i)}$ for some i, or $A \in \mathscr{D}(\pi)$ where π is a transposition, then $A \in \mathscr{A}$. As $E_{ij} \in \mathscr{A}$, then $F_{ij} = T(E_{ij})$ belongs to $\mathscr{A}$ too, and consequently $h(F_{ij}) \leqslant 3$.

LEMMA 6. *If $n = 3$, T is nonsingular, and $T(\mathscr{A}) \subseteq \mathscr{A}$, then $h(F_{ij}) = 1$.*

Proof. Assume that $h(F_{i_0 j_0}) = 2$ or $h(F_{i_0 j_0}) = 3$. Let Ω_0 be the set of positions where $F_{i_0 j_0}$ has a nonzero entry. Let (p, q), $(p'\, q') \in \Omega_0$ such that $(F_{i_0 j_0})_{pq} \neq 0$ and $(F_{i_0 j_0})_{p'q'} \neq 0$, and $\phi, \phi' \in A_3$ such that $\phi(p) = q$ and $\phi'(p') = q'$. Denote $\Omega_1 = D(\phi) \cup D(\phi') \setminus \{(p, q), (p', q')\}$.

As $F_{i_0 j_0} \in \mathscr{A}$, it has at most one nonzero entry in the ϕ-diagonal and one nonzero entry in the ϕ'-diagonal; because $\phi, \phi' \in A_3$, we have that $F_{i_0 j_0}$ has zero entries in Ω_1.

Now we show that if X belongs to $R_{(i_0)}$, $R^{(j_0)}$, or $D(\pi)$, where π is a transposition such that $\pi(i_0) = j_0$, then $T(X)$ has zero elements in the positions of Ω_1. Assume there is $(p_1, q_1) \in \Omega_1$ such that $T(X)_{p_1 q_1} \neq 0$. Then it is a simple exercise to show that we can choose complex numbers a and b so that the matrix $aF_{i_0 j_0} + bT(X)$ has two nonzero entries in $D(\phi)$ or in $D(\phi')$. But this matrix is the image under T of $aE_{i_0 j_0} + bX$, and so belongs to $\mathscr{A}$, a contradiction by Lemma 5.

Denote by Λ the set of positions covered by row i_0, column j_0, and the diagonal $D(\pi)$. Notice that (i_0, j_0) is common to row i_0, column j_0, and the diagonal π, because $\pi(i_0) = j_0$. It can be easily seen that $|\Lambda| = 7$. Then the dimension of the subspace generated by $\{E_{ij} : (i, j) \in \Lambda\}$ is 7.

We know that if $(i, j) \in \Lambda$, then $(F_{ij})pq = 0$ for all $(p, q) \in \Omega_1$. As $|\Omega_1| = 4$, this means that the dimension of the subspace generated by $\{F_{ij} : i, j \in \Lambda\}$ is at most 5, a contradiction, because T is nonsingular. ∎

Proof of Theorem 3. Let ψ be a permutation acting on the Cartesian product $\{1,2,3\} \times \{1,2,3\}$. By Lemma 6, and bearing in mind that T is nonsingular,

$$T(E_{ij}) = c_{\psi(i,j)} E_{\psi(i,j)},$$

where $\psi(i,j)$ is the image of (i,j) under ψ.

Let $\sigma_1, \sigma_2, \sigma_3$ be the three permutations of A_3, and let $\sigma_1', \sigma_2', \sigma_3' \in S_3 \setminus A_3$. We have $\chi(\sigma_i) \neq 0$ for $i = 1,2,3$ and $\chi(\sigma_i') = 0$ for $i = 1,2,3$.

For any i

$$d_\chi\left(\sum_j E_{j\sigma_i(j)}\right) = \chi(\sigma_i) \neq 0.$$

By Lemma 6, $T(\sum_j E_{j\sigma_i(j)})$ will have at most three nonzero entries. Since

$$d_\chi\left(T\left(\sum_j E_{j\sigma_i(j)}\right)\right) = d_\chi\left(\sum_j E_{j\sigma_i(j)}\right) \neq 0,$$

$T(\sum_j E_{j\sigma_i(j)})$ will have exactly three nonzero entries, and they must lie on a diagonal of type $(1, \sigma_k(1)), (2, \sigma_k(2)), (3, \sigma_k(3))$ for some k. Thus ψ induces a permutation $\bar{\psi}$ on A_3 $[\bar{\psi}(\sigma_i) = \sigma_k$, etc.]. Thus $T(E_{j\sigma_i(j)})$ will be one of the three $E_{h\bar{\psi}(\sigma_i)(h)}$ $(h \in \{1,2,3\})$ multiplied by a constant. We denote the value of h for which this is equal to $T(E_{j\sigma_i(j)})$ by $j_{\bar{\psi},i}$ or simply by j_i (since, once $\bar{\psi}$ has been fixed, it depends on σ_i and j only). Of course $j \to j_i$ (i fixed) is a permutation of $\{1,2,3\}$. Thus

$$T(E_{j\sigma_i(j)}) = c_{j_i\bar{\psi}(\sigma_i)(j_i)} E_{j_i\bar{\psi}(\sigma_i)(j_i)}.$$

Let X be an arbitrary matrix. It can be written

$$X = \sum_i \sum_j a_{j\sigma_i(j)} E_{j\sigma_i(j)}. \tag{4.7}$$

Therefore

$$T(X) = \sum_i \sum_j a_{j\sigma_i(j)} c_{j_i\bar{\psi}(\sigma_i)(j_i)} E_{j_i\bar{\psi}(\sigma_i)(j_i)} = C * Y.$$

Comparing with (4.7), we see that Y is obtained from X by (1) permuting the elements in each σ_i-diagonal (effect of $j \to j_i$), (2) permuting the σ_i-diagonals among themselves [effect of $\sigma_i \to \bar{\psi}(\sigma_i)$].

Now set $X = P(\sigma^{-1})$. We have

$$P(\sigma^{-1}) = \sum_j E_{j\sigma(j)}$$

and so

$$T(P(\sigma^{-1}) = C * \sum_j E_{j_i \bar{\psi}(\sigma_i)(j_i)} = C * \sum_j E_{j\bar{\psi}(\sigma_i)(j)}.$$

Since T preserves d_χ, we can write

$$\chi(\sigma) = \prod_j c_{j\bar{\psi}(\sigma)(j)} \chi(\bar{\psi}(\sigma)).$$

Conversely, it is easy to see that if T is defined as above it preserves the second immanant. ■

REMARK. Let $X = [x_{ij}]$ be 3×3 and

$$T(X) = \begin{bmatrix} \frac{1}{2} & -2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} x_{12} & x_{11} & x_{13} \\ x_{21} & x_{23} & x_{22} \\ x_{33} & x_{32} & x_{31} \end{bmatrix}. \tag{4.8}$$

The operator T preserves d_χ (χ is the nonlinear character of S_3). However, it is not possible to express $T(X)$ in a form similar to that of the case $n > 3$, because each column of the second matrix on the right-hand side of (4.8) includes elements from different columns of X.

NOTE. Professor G. de Oliveira observed that our Theorem 3 can also be obtained from a result (Lemma 6) of P. Botta in Linear transformations on matrices: The invariance of a class of general matrix functions, *Canad. J. Math.* 19:281–290 (1967).

The author wishes to thank Professor Graciano de Oliveira for suggesting this problem, for his guidance, and for his helpful remarks and suggestions.

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Received 3 December 1992; final manuscript accepted 9 June 1993